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Wave function in the presence of constraints: Persistent current in coupled rings

D. Schmeltzer
Physics Department, City College of the City University of New York, New York, New York 10031, USA

Avadh Saxena
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

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Using a method introduced earlier, we compute the wave function in the presence of constraints. As an explicit example we compute the wave function for the many electrons problem in coupled metallic rings in the presence of external magnetic fluxes. For equal fluxes and an even number of electrons the constraints enforce a wave function with a vanishing total momentum and a large persistent current and magnetization in contrast to the odd number of electrons where at finite temperatures the current is suppressed. We propose that the even-odd property can be verified by measuring the magnetization as a function of a varying gate voltage coupled to the rings. By reversing the flux in one of the rings the current and magnetization vanish in both rings; this can be potentially used as a nonlocal control device.

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I. INTRODUCTION

Recently a method for constraints has been introduced for computing the wave function for electronic systems by one of us (D.S.). The electronic wave function in quantum nanosystems at low temperatures is sensitive to interactions and topology such as the genus number g (Refs. 1 and 2) (the number of holes on a closed surface). As a result, the wave function has to satisfy certain *constraints*, which generate conserved currents.[3,4] The implementation of the constraints is a nontrivial task in Quantum Mechanics.[4] The root of the difficulty is that for a given constraint the Hermitian conjugate constraint operator might not be a constraint, therefore a reduction of the phase space is not possible.[4] This problem is solved by including nonphysical *ghost* fields.[5] In Classical Mechanics second class constraints[4] are solved by replacing the *Poisson* brackets by the *Dirac* brackets and quantization is performed according to the *Dirac* correspondence principle[4,5] with the undesirable feature that the quantum representation for the operators might not always be possible. Here, we will solve the constraints (using the method discussed by D.S.) without the need to introduce nonphysical operators.

As a model problem, we will consider the Aharonov-Bohm geometry[6–11] for the case that the genus is $g=2$. This corresponds to a double ring structure perfectly glued at one point to form a character "8" structure (see Fig. 1). Such a structure gives rise to an interesting quantum mechanical problem.[6] *Gluing* the two rings at the common point $x=0$ gives rise to a constraint problem, which was solved numerically using the *Dirac brackets*.[5]

In Sec. II, we present the method for computing the wave function with constraints. We will work with a folded geometry (i.e., the two rings on top of each other with a common point), therefore, the problem will be equivalent to a two-component spinor on a single ring. The constraint is such that at the common point $x=0$, L the annihilation operators are identified as a single operator $[C_1(x)-C_2(x)]|_{x=0,L}=0$.

Using the Dirac method,[3] we compute the Noether currents which allow us to identify the constraint currents. In the presence of external fluxes the constraints are translated into a set of equations for the wave function. The constraints induce correlations between the different components of the wave function. For noninteracting electrons the wave function for N electrons is given by the *Slater determinant* of the single particle states, but the current is the same if we sum over the single particle electronic states. For the present (interacting or correlated) problem we must work with the many-body wave function of the two rings (which is not a simple product of the two ring wave functions).

In Sec. III, we discuss the constraint method with the scattering theory and show that the strong coupling limit between the rings corresponds to the constraint problem considered in Sec. II. In Sec. IV, we use the constraint method to compute the many-body wave function for two rings in the presence of constraints. We find that the many particle wave function built from the single particle wave function which obeys the constraints is different from the many-body wave function which obeys the constraints. We show that the constraints impose additional relations between the amplitudes of the many-body wave function. In Sec. V, we present the modification needed in order to include the physical geometry of the rings, e.g., finite thickness. As a concrete example we choose two narrow cylinders which are in contact on the line $x=0$. Section VI is devoted to discussion and conclusions.

II. CONSTRAINT METHOD FOR TWO RINGS

We consider two rings threaded by a magnetic flux Φ_α, where $\alpha=1,2$ represents the index for each ring $\varphi_\alpha=2\pi(\frac{e\Phi_\alpha}{hc})=2\pi\frac{\Phi_\alpha}{\Phi_0}\equiv 2\pi\hat{\varphi}_\alpha$. The rings have a common point at $y=0$ (Fig. 1). The first ring is restricted to the region $0\leq y\leq L$ and the second ring is restricted to $-L\leq y\leq 0$. We introduce for the first ring $0\leq y\leq L$, $C_1(x)=C(x)=C(y)$, $C_1^\dagger(x)=C^\dagger(x)=C^\dagger(y)$, and for the second ring with $-L\leq y\leq 0$, $C_2(x)=C(-x)=C(y)$, and $C_2^\dagger(x)=C^\dagger(-x)=C^\dagger(y)$. The *two component spinor operator* $\hat{C}(x)\equiv[C_1(x),C_2(x)]$ with $0\leq x$

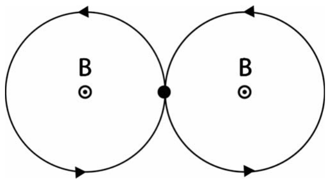

FIG. 1. Two coupled rings with a flux. The coordinate y is defined counterclockwise along the circumference of the rings with the common point at $y=0$. The left ring is restricted to the region $-L \leq y \leq 0$ and the right ring is restricted to the region $0 \leq y \leq L$.

$\leq L$ obeys periodic boundary conditions $\hat{C}(x)=\hat{C}(x+L)$ and $\hat{C}^{\dagger}(x)=\hat{C}^{\dagger}(x+L)$. Due to the folding, two equal fluxes $\hat{\varphi}_1 = \hat{\varphi}_2 \equiv \hat{\varphi}$ will be described by two opposite fluxes.

$$H=\int_0^L dx\left[\frac{\hbar^2}{2m}C_1^{\dagger}(x)\left(-i\partial_x-\frac{2\pi}{L}\hat{\varphi}_1\right)^2 C_1(x) + \frac{\hbar^2}{2m}C_2^{\dagger}(x)\left(i\partial_x-\frac{2\pi}{L}\hat{\varphi}_2\right)^2 C_2(x)\right]. \quad (1)$$

A. Continuity constraint

The Hamiltonian given in Eq. (1) will be investigated under the condition that the annihilation operators at the contact point must be identified as one operator. This is implemented with the help of the constraint operator,

$$\eta \equiv [C_1(x)-C_2(x)]|_{x=0,L}; \quad \eta|\chi,N\rangle = 0, \quad (2)$$

where, $|\chi,N\rangle$ denotes the eigenstate for N particles. Following Dirac,[3] the constraints and the time derivative of the constraints must be satisfied at any time. We must have $\eta|\chi,N\rangle=0$ and $\frac{d}{dt}\eta|\chi,N\rangle=0$.

B. Eigenvalue constraint

In order to satisfy the constraint at any time we need to show that $\frac{d}{dt}\eta|\chi,N\rangle=0$. This equation is determined by the time evolution of the constraint operator η. Following Dirac[3] we introduce a Lagrange multiplier λ and replace the Hamiltonian H by the total Hamiltonian:[3] $H_T=H+\lambda\eta^{\dagger}\eta$, where η is the constraint. Using the Heisenberg equation of motion we obtain:

$$\begin{aligned}\frac{d}{dt}\eta|\chi,N\rangle &= \frac{1}{i\hbar}[\eta,H_T]|\chi,N\rangle \\ &= \frac{1}{i\hbar}([\eta,H]-2\lambda\eta^{\dagger}\eta+2\lambda\eta)|\chi,N\rangle \\ &= \frac{1}{i\hbar}[\eta,H]|\chi,N\rangle \\ &= 0.\end{aligned}$$

In obtaining this result we have used the relations: $[\eta,\lambda\eta^{\dagger}\eta]|\chi,N\rangle=(-2\lambda\eta^{\dagger}\eta+2\lambda\eta)|\chi,N\rangle=0$ with the constraint condition $\eta|\chi,N\rangle=0$. As a result we find that the condition $\frac{d}{dt}\eta|\chi,N\rangle=0$ generates a constraint operator which we identify as the eigenvalue constraint operator E given by $[\eta,H]\equiv\frac{\hbar^2}{2m}E$:

$$E \equiv \left[\left(-i\partial_x-\frac{2\pi}{L}\hat{\varphi}_1\right)^2 C_1(x) - \left(-i\partial_x+\frac{2\pi}{L}\hat{\varphi}_2\right)^2 C_2(x)\right]\Big|_{x=0,L}; \quad E|\chi,N\rangle=0. \quad (3)$$

C. Current constraint: The Noether current for a periodic two component spinor system

Using the periodicity of the two component spinor $\hat{C}^{\dagger}(x)=\hat{C}^{\dagger}(x+L)$ we perform periodic gauge transformations $\epsilon(x)=\epsilon(x+L)$ and compute the Noether current. The infinitesimal gauge transformation for the two component spinor $\hat{C}^{\dagger}(x)$, $\tilde{\hat{C}}^{\dagger}(x)|0\rangle=e^{i\epsilon(x)}\hat{C}^{\dagger}(x)|0\rangle$ takes the form in the component representation: $\tilde{C}_1^{\dagger}(x)|0\rangle=e^{i\epsilon(x)}C_1^{\dagger}(x)|0\rangle$ and $\tilde{C}_2^{\dagger}(x)|0\rangle=e^{i\epsilon(-x)}C_2^{\dagger}(x)|0\rangle$. The replacement of $e^{i\epsilon(x)}$ by $e^{i\epsilon(-x)}$ in the second equation is due to the folding transformation. As a result of the transformation the Hamiltonian $h=\frac{\hbar^2}{2m}[\delta_{\alpha,1}(-i\partial_x-\frac{2\pi}{L}\hat{\varphi}_1)^2+\delta_{\alpha,2}(-i\partial_x+\frac{2\pi}{L}\hat{\varphi}_2)^2]$ is replaced by

$$\tilde{h} \equiv \frac{\hbar^2}{2m}\left[\delta_{\alpha,1}\left\{-i\partial_x-\frac{2\pi}{L}\hat{\varphi}_1+\partial_x[\epsilon(x)]\right\}^2 + \delta_{\alpha,2}\left\{-i\partial_x+\frac{2\pi}{L}\hat{\varphi}_2+\partial_{-x}[\epsilon(-x)]\right\}^2\right].$$

The constraint is invariant under the gauge transformation $\eta^{\dagger}(x)\eta(x)=\tilde{\eta}^{\dagger}(x)\tilde{\eta}(x)$. The constraint operator η is replaced by the transformed one $\tilde{\eta}\equiv[e^{-i\epsilon(x)}\eta(x)]|_{x=0,L}\equiv[e^{-i\epsilon(x)}\tilde{C}_1(x) - e^{-i\epsilon(-x)}\tilde{C}_2(x)]|_{x=0,L}$, $\tilde{\eta}|\chi,N\rangle=0$. [$\epsilon(x)$ is an arbitrary periodic function in L, which is continuous at $x=0$ and has a continuous derivative $\partial_x[\epsilon(x)]\neq 0$ at $x=0$. For example, any function with the Fourier expansion $\epsilon(x)=\sum_{r=1}^{r=\infty}\hat{\epsilon}_r \sin[\frac{2\pi r}{L}x]$ and Fourier components $\sum_{r=1}^{r=\infty}\hat{\epsilon}_r\neq 0$ obeys this condition.] The transformed constraint $\tilde{\eta}|\chi,N\rangle=0$ must hold at any time, therefore we have the equation: $\frac{d}{dt}\tilde{\eta}|\chi,N\rangle=0$. Applying the Heisenberg equation of motion for the transformed Hamiltonian $\tilde{h}$ and keeping only first order terms in $\partial_x[\epsilon(x)]$, that obey $\partial_x[\epsilon(x)]|_{x=0}\neq 0$, gives us:

$$\begin{aligned} i\hbar\frac{d}{dt}\tilde{\eta}|\chi,N\rangle &= \frac{\hbar^2}{2m}\int_0^L dx\left[\tilde{\eta},\tilde{C}_1^{\dagger}(x)\left\{-i\partial_x-\frac{2\pi}{L}\hat{\varphi}_1+\partial_x[\epsilon(x)]\right\}^2\tilde{C}_1(x) + \tilde{C}_2^{\dagger}(x)\left\{-i\partial_x+\frac{2\pi}{L}\hat{\varphi}_2+\partial_{-x}[\epsilon(-x)]\right\}^2\tilde{C}_2(x)\right]|\chi,N\rangle \\ &= 0. \end{aligned} \quad (4)$$

Using the energy constraint $E|\chi,N\rangle=0$ we identify the *current* continuity constraint β:

$$\beta = \left[\left(-i\partial_x - \frac{2\pi}{L}\hat{\varphi}_1 \right) C_1(x) + \left(-i\partial_x + \frac{2\pi}{L}\hat{\varphi}_2 \right) C_2(x) \right]\Big|_{x=0,L}; \quad \beta|\chi,N\rangle = 0. \quad (5)$$

To conclude, the eigenstate $|\chi,N\rangle$ for N particles in two rings must satisfy the following equations:

$$H|\chi,N\rangle = E(N)|\chi,N\rangle; \quad \eta|\chi,N\rangle = 0; \quad E|\chi,N\rangle = 0;$$
$$\beta|\chi,N\rangle = 0. \quad (6)$$

The eigenfunctions will be given in terms of the amplitudes of the vector $|\chi,N\rangle$. For example the single particle state is given by:

$|\chi,N=1\rangle = \int_0^L dx[f_1(x)C_1^\dagger(x) + f_2(x)C_2^\dagger(x)]|0\rangle$. Similarly the two particle state is given by:

$$|\chi,N=2\rangle = \int_0^L dx \int_0^L dy[f_{1,1}(x,y)C_1^\dagger(x)C_1^\dagger(y) + f_{1,2}(x,y)C_1^\dagger(x)C_2^\dagger(y) + f_{2,2}(x,y)C_2^\dagger(x)C_2^\dagger(y)]|0\rangle. \quad (7)$$

The amplitudes $f_1(x)$, $f_2(x)$ and $f_{1,1}(x,y)$, $f_{1,2}(x,y)$, and $f_{2,2}(x,y)$ are determined by the condition given below in Eq. (8).

D. Current operator

The N particle wave function $\langle x_N, \ldots x_1|\chi,N\rangle$ must obey periodic boundary conditions:

$$\langle 0|C_{\alpha_1}(x_1)\,.\,.\,C_{\alpha_k}(x_k)\,.\,.\,C_{\alpha_N}(x_N)|\chi,N\rangle = \langle 0|C_{\alpha_1}(x_1)\,.\,.\,C_{\alpha_k}(x_k+L)\,.\,.\,C_{\alpha_N}(x_N)|\chi,N\rangle$$

where α_i takes two values $\alpha_i=1$ or $\alpha_i=2$.

Once the eigenfunction $|\chi,N\rangle$ is known we can use the current operators $\hat{J}_1(x)$ and $\hat{J}_2(x)$ in the second quantized form to compute the current in each ring:

$$\hat{J}_1(x) = \frac{\hbar}{i2m}\left\{ C_1^\dagger(x)\left(\partial_x - i\frac{2\pi}{L}\hat{\varphi}_1\right)C_1(x) - \left[\left(\partial_x + i\frac{2\pi}{L}\hat{\varphi}_1\right)C_1^\dagger(x)\right]C_1(x)\right\};$$

$$J_1(x) = \frac{\langle N,\chi|\hat{J}_1(x)|\chi,N\rangle}{\langle N,\chi|\chi,N\rangle},$$

$$\hat{J}_2(x) = \frac{\hbar}{i2m}\left\{ C_2^\dagger(x)\left(\partial_x + i\frac{2\pi}{L}\hat{\varphi}_2\right)C_2(x) - \left[\left(\partial_x - i\frac{2\pi}{L}\hat{\varphi}_2\right)C_2^\dagger(x)\right]C_2(x)\right\};$$

$$J_2(x) = \frac{\langle N,\chi|\hat{J}_2(x)|\chi,N\rangle}{\langle N,\chi|\chi,N\rangle}. \quad (8)$$

III. EMERGING CONSTRAINT CONDITIONS FROM THE TIGHT BINDING FORMULATION

The Hamiltonian in Eq. (1) must be supplemented by the coupling term between the rings. The most general form for the coupling is given by:

$$H_{\text{coupling}} = \int_0^L dx\,\delta(x)\{-U_\perp[C_1^\dagger(x)C_2(x) + C_2^\dagger(x)C_1(x)] + U_\parallel[C_1^\dagger(x)C_1(x) + C_2^\dagger(x)C_2(x)]\}$$
$$= \int_0^L dx\,\delta(x)\{U_\perp[C_1^\dagger(x) - C_2^\dagger(x)][C_1(x) - C_2(x)] + (U_\parallel - U_\perp)[C_1^\dagger(x)C_1(x) + C_2^\dagger(x)C_2(x)]\}. \quad (9)$$

We introduce the notation $U_\perp = tU$ and $U_\parallel = sU$ where s and t are parameters. Using the spinor representation $\hat{C}(x) = [C_1(x), C_2(x)]^T$ we can rewrite the coupling Hamiltonian in terms of the Pauli matrix σ_1 and the identity matrix I:

$$H_{\text{coupling}} = \int_0^L dx\,\delta(x)U\hat{C}^+(x)[(sI - t\sigma_1)]\hat{C}(x).$$

This problem belongs to the class of delta function potentials considered in Quantum Mechanics.

A. Wave function for a single particle, $N=1$

$$|\chi,N=1\rangle = \int_0^L dx[f_1(x)C_1^\dagger(x) + f_2(x)C_2^\dagger(x)]|0\rangle,$$

$$(H + H_{\text{coupling}})|\chi,N=1\rangle = E(1)|\chi,N=1\rangle.$$

As a result we obtain the Schrödinger equation in terms of the two amplitudes $f_1(x)$, $f_2(x)$. The Hamiltonian in Eq. (1) together with H_{coupling} can be solved using the method for delta function potentials. We integrate the single particle Schrödinger equation around $x=0$, L and obtain the discontinuity *derivative* of the spinor $\Psi(x) = [f_1(x), f_2(x)]^T$, which obeys $\frac{d\Psi(x)}{dx}\big|_{x=-\epsilon}^{x=\epsilon} \equiv \frac{d\Psi(x)}{dx}\big|_{x=L-\epsilon}^{x=\epsilon}$.

$$\left(-i\partial_x - \frac{4\pi}{L}\hat{\varphi}_1\right)f_1(x)\Big|_{x=L-\epsilon}^{x=\epsilon} = \frac{-i2m}{\hbar^2}U\frac{1}{2}\{[sf_1(\epsilon) - tf_2(\epsilon)] + [sf_1(L-\epsilon) - tf_2(L-\epsilon)]\},$$

$$\left(-i\partial_x + \frac{4\pi}{L}\hat{\varphi}_2\right)f_2(x)\Big|_{x=L-\epsilon}^{x=\epsilon} = \frac{-i2m}{\hbar^2}U\frac{1}{2}\{[sf_2(\epsilon) - tf_1(\epsilon)] + [sf_2(L-\epsilon) - tf_1(L-\epsilon)]\}.$$

This set of equations gives us the boundary conditions for the present problem. Indeed these equations are determined by the discontinuity function $U[sf_2(0) - tf_1(0)]$. For this case the solution follows from the method of the delta function potentials.

B. Wave function for two particles, $N=2$

In order to compute the wave function for N particles we have to compute the boundary conditions for the amplitudes of the wave function. We will consider the case of two particles which can be generalized to many particles.

$$|\chi,N=2\rangle = \int_0^L dx_1 \int_0^L dx_2 [f_{1,1}(x_1,x_2)C_1^\dagger(x_1)C_1^\dagger(x_2) + f_{1,2}(x_1,x_2)C_1^\dagger(x_1)C_2^\dagger(x_2) + f_{2,2}(x_1,x_2)C_2^\dagger(x_1)C_2^\dagger(x_2)] \times|0\rangle.$$

Using the eigenvalue equation: $(H+H_{\text{coupling}})|\chi,N=2\rangle =E(2)|\chi,N=2\rangle$ we integrate the two particle Schrödinger equation around $x_1=0, L$ and obtain the discontinuity *derivative* for the three amplitudes $f_{1,1}(x_1,x_2)$, $f_{2,2}(x_1,x_2)$, $f_{1,2}(x_1,x_2)$

$$\left(-i\partial_{x_1} - \frac{4\pi}{L}\hat{\varphi}_1\right) f_{1,1}(x_1,x_2)|_{x_1=L-\epsilon}^{x_1=\epsilon} = \frac{-i2m}{\hbar^2}U\frac{1}{2}[s(f_{1,1}(x_1=\epsilon,x_2)+f_{1,1}(x_1=L-\epsilon,x_2))],$$

$$\left(-i\partial_{x_1} - \frac{4\pi}{L}\hat{\varphi}_1\right) f_{2,2}(x_1,x_2)|_{x_1=L-\epsilon}^{x_1=\epsilon} = \frac{-i2m}{\hbar^2}U\frac{1}{2}[-t(f_{2,2}(x_1=\epsilon,x_2)+f_{2,2}(x_1=L-\epsilon,x_2))],$$

$$\left(-i\partial_{x_1} - \frac{4\pi}{L}\hat{\varphi}_1\right) f_{1,2}(x,x_2)|_{x_1=L-\epsilon}^{x_1=\epsilon} = \frac{-i2m}{\hbar^2}U\frac{1}{2}[(s-t)(f_{1,2}(x_1=\epsilon,x_2)+f_{1,2}(x_1=L-\epsilon,x_2))].$$

Similar equations are obtained by exchanging x_1 with x_2. This set of equations determines the two particle wave function $\langle x_1,x_2|\chi,N=2\rangle$. This procedure is rather involved but can be generalized to the N particles case.

C. Strong coupling limit $U\to\infty$

Next we investigate the strong coupling limit and show that the problem can be simplified to a constraint problem. We consider the case $s=t=1$ for which we have the scattering matrix S given by:

$$S = T\exp\left(-i\tfrac{U}{\hbar}\int_{-\infty}^{\infty}d\tau\{[C_1^+(x=0,\tau)-C_2^+(x=0,\tau)] \times[C_1(x=0,\tau)-C_2(x=0,\tau)]\}\right).$$

For $U\to\infty$ the scattering matrix S obeys:

$$\lim_{U\to\infty} T\exp\left(-i\frac{U}{\hbar}\int_{-\infty}^{\infty}d\tau\{[C_1^+(x=0,\tau)-C_2^+(x=0,\tau)] \times[C_1(x=0,\tau)-C_2(x=0,\tau)]\}\right)|\chi,N\rangle \to[C_1(x=0,\tau)-C_2(x=0,\tau)]|\chi,N\rangle=0.$$

As a result the field $[C_1(x=0,\tau)-C_2(x=0,\tau)]$ is enforced to satisfy $C_1(x=0,\tau)-C_2(x=0,\tau)=0$, which is equivalent to the constraint condition: $\eta|\chi,N\rangle\equiv[C_1(x)-C_2(x)]|_{x=0,L}|\chi,N\rangle=0$.

IV. COMPUTATION OF THE WAVE FUNCTION FOR EQUAL FLUXES

For the strong coupling limit we will use the constraints given by Eq. (6). When the fluxes are the same for both rings the constraint operator β is simplified to a constraint $\gamma =i\beta(\hat{\varphi}_1=\hat{\varphi}_2)$:

$$\gamma=[\partial_x C_1(x)+\partial_x C_2(x)]|_{x=0,L};\quad \gamma|\chi,N\rangle=0. \tag{10}$$

The N particles wave function for equal fluxes must satisfy the following conditions:

$$H|\chi,N\rangle=E(N)|\chi,N\rangle;\quad \eta|\chi,N\rangle=0;\quad E|\chi,N\rangle=0;$$
$$\gamma|\chi,N\rangle=0. \tag{11}$$

A. Single particle state

The *single* particle case corresponds to one electron in two rings. The state for one particle is given by: $|\chi,N=1\rangle =\int_0^L dx[f_1(x)C_1^\dagger(x)+f_2(x)C_2^\dagger(x)]|0\rangle$. The two-component spinor amplitudes $f_1(x)$ and $f_2(x)$ represent the wave function. Using the Hamiltonian given in Eq. (1) we can write down the eigenvalue equation $H|\chi,N=1\rangle=E(1)|\chi,N=1\rangle$. A standard calculation shows this equation is equivalent to two eigenvalue equations for the amplitudes $f_1(x)$ and $f_2(x)$.

$$\frac{\hbar^2}{2m}\left(-i\partial_x - \frac{2\pi}{L}\hat{\varphi}\right)^2 f_1(x)=E(1)f_1(x);$$
$$\frac{\hbar^2}{2m}\left(-i\partial_x + \frac{2\pi}{L}\hat{\varphi}\right)^2 f_2(x)=E(1)f_2(x). \tag{12}$$

The constraint operators given in Eq. (8) generate the following boundary conditions at $x=0$:

$$f_1(x=0)=f_2(x=0);\quad [\partial_x f_1(x)+\partial_x f_2(x)]|_{x=0}=0. \tag{13}$$

The first equation is equivalent to the continuity of the wave function at $x=0$ and the second equation describes the continuity of the derivative of the wave function (once we fold back the space) at $x=0$. From the eigenvalue equation given in Eq. (12) we find: $E(n;N=1)=\frac{\hbar^2}{2m}(\frac{2\pi}{L})^2(n-\hat{\varphi})^2$ for the ring one and $E(-n;N=1)=\frac{\hbar^2}{2m}(\frac{2\pi}{L})^2(n-\hat{\varphi})^2$ for the second ring. Due to the folding of the space around $x=0$, the eigenvalue with the quantum number n in ring one and quantum number $-n$ in the second ring are equal. This result holds for the quantum numbers, $n=0,\pm1,\pm2,\ldots$. The single particle state $|n,N=1\rangle$ for $\hat{\varphi}\neq\frac{1}{2}$ is given by:

$$|n;N=1\rangle=\frac{1}{\sqrt{2L}}\int_0^L dx[e^{i(2\pi/L)nx}C_1^\dagger(x)+e^{-i(2\pi/L)nx}C_2^\dagger(x)]|0\rangle. \tag{14}$$

To understand this result we fold back the ring such that $x\to-x$. This means that if the particle in the first ring (x

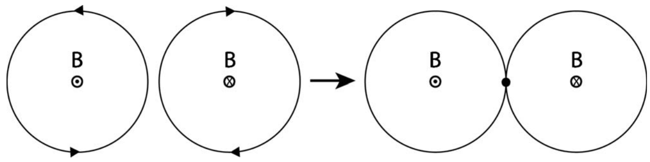

FIG. 2. Two uncoupled and coupled rings with opposite flux. The coordinate y is defined along the rings as in Fig. 1.

<0) has the momentum $\frac{2\pi}{L}n$ it will be perfectly transmitted to the second ring with the same momentum and the same amplitude. If we remove the point $x=0$ and create a ring of double length $2L$, the current will be the same as in one ring with the same flux. Indeed, the only difference being the doubling of the size. As a result, we will have half of the current in a single ring. (If we rescale the length, we find the same current as in one ring[5]) It is important to remark that the states $|n;N=1\rangle$ and $|-n;N=1\rangle$ correspond to two different eigenvalues. Therefore, for a given eigenvalue we cannot have a linear combination of waves $e^{i(2\pi/L)nx}$ and $e^{-i(2\pi/L)nx}$ in the same ring. The wave $e^{i(2\pi/L)nx}$ in ring one will be transmitted into the second ring without any reflection, the form of the transmitted wave will be $e^{-i(2\pi/L)nx}$ (in the unfolded coordinates the form of the wave will be $e^{i(2\pi/L)ny}$ in the second ring for $y<0$). In Fig. 1, we show the current flow for two rings with equal fluxes in the unfolded geometry. The current vanishes if we have the opposite flux in the two rings, as depicted in Fig. 2 and discussed in Sec. V.

The case $\hat{\varphi}=\frac{1}{2}$ deserves special consideration. The eigenvalue operator E has two pairs of momentum with the same eigenvalue: The first pair $n_1=n$ in the first ring and $n_2=-n$ for the second ring and the second pair $n_1'=-n+2\hat{\varphi}$ (ring one) and $n_2'=n-2\hat{\varphi}$ (ring two). Consequently, we obtain two degenerate eigenstates $|n;N=1,+\rangle$ and $|n;N=1,-\rangle$ given by:

$$|n;N=1,+\rangle=\frac{1}{\sqrt{2L}}\int_0^L dx[e^{i(2\pi/L)nx}C_1^\dagger(x)+e^{-i(2\pi/L)nx}C_2^\dagger(x)]\times|0\rangle;$$

$$|n;N=1,-\rangle=\frac{1}{\sqrt{2L}}\int_0^L dx[e^{-i(2\pi/L)(n-2\hat{\varphi})x}C_1^\dagger(x)+e^{i(2\pi/L)(n-2\hat{\varphi})x}C_2^\dagger(x)]\times|0\rangle. \quad (15)$$

As a result the current for the state $|n;N=1,-\rangle$ will be opposite to the current for the state $|n;N=1,+\rangle$ shown in Fig. 1. Since the two eigenstates $|n;N=1,+\rangle$ and $|n;N=1,-\rangle$ are degenerate, the single particle state will be given by two linear combinations of the eigenstates $|n;N=1,+\rangle$ and $|n;N=1,-\rangle$: $|\chi(n),\hat{\varphi}=\frac{1}{2};N=1\rangle=\alpha_+|n;N=1,+\rangle\pm\alpha_-|n;N=1,-\rangle$ with the condition $|\alpha_+|^2+|\alpha_-|^2=1$. For the special values $|\alpha_+|^2=|\alpha_-|^2$ the current will vanish.

B. Two particles state

We will construct the two particles state and show that due to the constraints not all the antisymmetric combinations of the single particle states which obey the constraints are allowed. Imposing the constraints on the two particles state imposes further restrictions. The two particles eigenstate is determined by the three components $f_{11}(x_1,x_2)$, $f_{12}(x_1,x_2)$, and $f_{22}(x_1,x_2)$ that obey the eigenvalue equations:

$$\frac{\hbar^2}{2m}\left[\left(-i\partial_{x_1}-\frac{2\pi}{L}\hat{\varphi}\right)^2+\left(-i\partial_{x_2}-\frac{2\pi}{L}\hat{\varphi}\right)^2\right]f_{11}(x_1,x_2)=E(2)f_{11}(x_1,x_2),$$

$$\frac{\hbar^2}{2m}\left[\left(-i\partial_{x_1}-\frac{2\pi}{L}\hat{\varphi}\right)^2+\left(i\partial_{x_2}-\frac{2\pi}{L}\hat{\varphi}\right)^2\right]f_{12}(x_1,x_2)=E(2)f_{12}(x_1,x_2),$$

$$\frac{\hbar^2}{2m}\left[\left(i\partial_{x_1}-\frac{2\pi}{L}\hat{\varphi}\right)^2+\left(i\partial_{x_2}-\frac{2\pi}{L}\hat{\varphi}\right)^2\right]f_{22}(x_1,x_2)=E(2)f_{22}(x_1,x_2).$$

The amplitudes $f_{11}(x_1,x_2)$, $f_{12}(x_1,x_2)$, and $f_{22}(x_1,x_2)$ are constructed from the single particle states which are represented in terms of the complex coordinate $Z(x)=e^{i(2\pi/L)x}$ and $Z^*(x)=e^{-i(2\pi/L)x}$. We introduce the antisymmetry operator $\tilde{\mathcal{A}}$, which acts both on the space coordinates and the ring index matrices A_{11} (two particles on ring one), A_{12} (one particle on ring one and the second on ring two), and A_{22} (two particles on ring two). When the operator $\tilde{\mathcal{A}}$ acts on a two particle wave function it gives:

$$\tilde{\mathcal{A}}[A_{12}(Z(x_1))^n(Z(x_2))^m]\equiv[A_{12}(Z(x_1))^m(Z(x_2))^n-A_{21}(Z(x_2))^n(Z(x_1))^m]$$

and

$$\tilde{\mathcal{A}}[A_{ii}(Z(x_1))^n(Z(x_2))^m]\equiv[A_{ii}(Z(x_1))^m(Z(x_2))^n-A_{ii}(Z(x_2))^n(Z(x_1))^m]$$

for $i=1,2$.

From the eigenvalue constraint $E|n,m;N=2\rangle=0$ we obtain the condition for the eigenvalues. The only possible solution for these equations is states with $m=-n$ which give eigenvalues

$$E(2)=E(n,-n;N=2)=\tfrac{\hbar^2}{2m}\left(\tfrac{2\pi}{L}\right)^2[(n-\hat{\varphi})^2+(-n-\hat{\varphi})^2,]$$

$n=0,\pm1,\pm2\ldots$

For amplitude $f_{11}(x_1,x_2)$ we consider only the single particle states with n and $-n$ which have the eigenvalue $E(2)=\frac{\hbar^2}{2m}(\frac{2\pi}{L})^2[(n-\hat{\varphi})^2+(-n-\hat{\varphi})^2]$. We construct the antisymmetric amplitudes which are given by:

$$f_{11}(x_1,x_2)=A_{11}[(Z(x_1))^n(Z(x_2))^{-n}-(Z(x_2))^n(Z(x_1))^{-n}].$$

Similarly for two electrons on the second ring $f_{22}(x_1,x_2)$ we have:

$$f_{22}(x_1,x_2)=B_{11}[(Z(x_1))^{-n}(Z(x_2))^n-(Z(x_2))^{-n}(Z(x_1))^n].$$

The amplitude for one electron on ring one and the second electron on ring two is given by $f_{12}(x_1,x_2)$. This corresponds to two pairs of states n, n and $-n$, and $-n$. The eigenvalue for the pair n, n is equal to $E(2)=\frac{\hbar^2}{2m}(\frac{2\pi}{L})^2[(n-\hat{\varphi})^2+(-n-\hat{\varphi})^2]$. For $-n$, $-n$ we have the same eigenvalue. The amplitude $f_{12}(x_1,x_2)$ is given by a linear combination of the two pairs. Using the antisymmetry operator $\tilde{\mathcal{A}}$ we obtain the amplitude $f_{12}(x_1,x_2)$ for the two pairs:

$$\begin{aligned} f_{12}(x_1,x_2) &= \tilde{\mathcal{A}}\{A_{12}[Z(x_1)]^n[Z(x_2)]^n\} \\ &\quad + \tilde{\mathcal{A}}\{B_{12}[Z(x_1)]^{-n}[Z(x_2)]^{-n}\} \\ &= \{A_{12}[Z(x_1)]^n[Z(x_2)]^n - A_{21}[Z(x_2)]^n[Z(x_1)]^n\} \\ &\quad + \{B_{12}[Z(x_1)]^{-n}[Z(x_2)]^{-n} \\ &\quad - B_{21}[Z(x_2)]^{-n}[Z(x_1)]^{-n}\}. \end{aligned}$$

Using constraints given in Eq. (8) for the two particles state $|n,m;N=2\rangle$: $\eta|n,m;N=2\rangle=0$, $E|n,m;N=2\rangle=0$, and $\gamma|n,m;N=2\rangle=0$, we obtain the following boundary conditions:

$$2f_{11}(x_1,0) = f_{1,2}(x_1,0);$$

$$[2\partial_{x_2}f_{11}(x_1,x_2) + \partial_{x_2}f_{12}(x_1,x_2)]_{x_2=0} = 0,$$

$$2f_{2,2}(x_1,0) = f_{1,2}(0,x_1);$$

$$[2\partial_{x_2}f_{22}(x_2,x_1) + \partial_{x_2}f_{12}(x_1,x_2)]_{x_2=0} = 0.$$

From these equations we find that the amplitudes obey the relations: $A_{12}=-A_{21}=2A_{11}$; $A_{11}=A_{22}$ and $B_{21}=-B_{12}=2A_{22}$. We introduce the antisymmetric spinor notation $\epsilon_{1;2}\equiv\frac{A_{12}}{2}$, which obeys the relations: $\epsilon^{1,1}_{1;2}=-\epsilon^{1,1}_{2;1}$ and $(\epsilon^{1,1}_{1;2})^\dagger\cdot\epsilon^{1,1}_{1;2}=1$ (the upper index 1,1 means that we have one electron in each ring, the bottom index 1,2 or 2,1 represents the order). For example, $\epsilon^{1;1}_{1;2}$ denotes the first electron is on ring one and the second electron is on ring two and $\epsilon^{1;1}_{2;1}$ represents the first electron is on ring two and the second electron is on ring one. The normalized two particle state is given by:

$$\begin{aligned} |n,-n;N=2\rangle &= \int_0^L dx_1\int_0^L dx_2[f_{11}(x_1,x_2)C_1^\dagger(x_1)C_1^\dagger(x_2) \\ &\quad + f_{12}(x_1,x_2)C_1^\dagger(x_1)C_2^\dagger(x_2) \\ &\quad + f_{22}(x_1,x_2)C_2^\dagger(x_1)C_2^\dagger(x_2)]|0\rangle \\ &= \int_0^L dx_1\int_0^L dx_2\frac{1}{4L}[\{[Z(x_1)]^n[Z^*(x_2)]^n \\ &\quad - [Z(x_2)]^n[Z^*(x_1)]^n\}C_1^\dagger(x_1)C_1^\dagger(x_2) \\ &\quad + 2\epsilon^{1;1}_{1;2}\{[Z(x_1)]^n[Z(x_2)]^n \\ &\quad - [Z^*(x_2)]^n[Z^*(x_1)]^n\}C_1^\dagger(x_1)C_2^\dagger(x_2) \\ &\quad + \{[Z^*(x_1)]^n[Z(x_2)]^n \\ &\quad - [Z^*(x_2)]^n[Z(x_1)]^n\}C_2^\dagger(x_1)C_2^\dagger(x_2)]|0\rangle. \end{aligned} \tag{16}$$

The off-diagonal spinor component $f_{12}(x_1,x_2) \propto 4i\sin\{\frac{2\pi}{L}n[(x_1+x_2)]\}$ is symmetric in space and resembles the *BCS* pairing wave function (once we identify the ring index with the spin) in contrast to the diagonal elements $f_{11}(x_1,x_2)$ and $f_{22}(x_1,x_2)$, which are antisymmetric in space. The two particles state, which obeys the constraints, is different from the two particles state constructed from the single particle states, which obey the constraints. Using the single particle states $|n;N=1\rangle$ and $|m;N=1\rangle$ [which obey Eq. (11)] we construct an antisymmetric tensor product $|n,m;N=2\rangle_{\text{build}}=|n;N=1\rangle|m;N=1\rangle-|m;N=1\rangle|n;N=1\rangle$. This state is not a solution, which obeys the constraints for the two particles state. The only possibility is to have an antisymmetric tensor product of two states with vanishing total momentum $|n,-n;N=2\rangle=|n;N=1\rangle|-n;N=1\rangle-|-n;N=1\rangle|n;N=1\rangle$. [The ground state for the two particles ($\hat{\varphi}<\frac{1}{2}$) is given by the eigenstate $|1,-1;N=2\rangle$.] This structure persists for an even number of electrons $N=2M$ and gives rise to a robust state absent for the single ring.

C. Three particles state

The wave function for *three* particles can only be found for special configurations $|m,n,-n;N=3\rangle$, $m\neq n$ and $m\neq -n$. The ground state will be given by the state $|0,1,-1;N=3\rangle$. The three particles state is determined by the four amplitudes $f_{111}(x_1,x_2,x_3)$, $f_{112}(x_1,x_2,x_3)$, $f_{122}(x_1,x_2,x_3)$, and $f_{222}(x_1,x_2,x_3)$, which obey the eigenvalue equation:

$$\begin{aligned} &\frac{\hbar^2}{2m}\left[\left(-i\partial_{x_1}-\frac{2\pi}{L}\hat{\varphi}\right)^2 \right. \\ &\quad \left. + \left(-i\partial_{x_2}-\frac{2\pi}{L}\hat{\varphi}\right)^2 + \left(-i\partial_{x_3}-\frac{2\pi}{L}\hat{\varphi}\right)^2\right]f_{111}(x_1,x_2,x_3) \\ &= E(3)f_{111}(x_1,x_2,x_3), \end{aligned}$$

$$\begin{aligned} &\frac{\hbar^2}{2m}\left[\left(-i\partial_{x_1}-\frac{2\pi}{L}\hat{\varphi}\right)^2 + \left(-i\partial_{x_2}-\frac{2\pi}{L}\hat{\varphi}\right)^2 \right. \\ &\quad \left. + \left(-i\partial_{x_3}+\frac{2\pi}{L}\hat{\varphi}\right)^2\right]f_{112}(x_1,x_2,x_3) \\ &= E(3)f_{112}(x_1,x_2,x_3), \end{aligned}$$

$$\begin{aligned} &\frac{\hbar^2}{2m}\left[\left(-i\partial_{x_1}-\frac{2\pi}{L}\hat{\varphi}\right)^2 + \left(-i\partial_{x_2}+\frac{2\pi}{L}\hat{\varphi}\right)^2 \right. \\ &\quad \left. + \left(-i\partial_{x_3}+\frac{2\pi}{L}\hat{\varphi}\right)^2\right]f_{122}(x_1,x_2,x_3) \\ &= E(3)f_{122}(x_1,x_2,x_3), \end{aligned}$$

$$\begin{aligned} &\frac{\hbar^2}{2m}\left[\left(-i\partial_{x_1}+\frac{2\pi}{L}\hat{\varphi}\right)^2 + \left(-i\partial_{x_2}+\frac{2\pi}{L}\hat{\varphi}\right)^2 \right. \\ &\quad \left. + \left(-i\partial_{x_3}+\frac{2\pi}{L}\hat{\varphi}\right)^2\right]f_{222}(x_1,x_2,x_3) \\ &= E(3)f_{222}(x_1,x_2,x_3). \end{aligned}$$

Using Eq. (8) we obtain the following relations for the spinor components:

$$3f_{111}(x_1,x_2,0)=f_{112}(x_1,x_2,0);$$

$$3f_{222}(0,x_1,x_2)=f_{122}(0,x_1,x_2);$$

$$[3\partial_{x_3}f_{222}(x_3,x_1,x_2)+\partial_{x_3}f_{112}(x_2,x_1,x_3)]_{x_3=0}=0,$$

$$2f_{121}(x_1,x_2,0)=f_{122}(x_2,x_1,0);$$

$$[3\partial_{x_3}f_{121}(x_1,x_2,x_3)+\partial_{x_3}f_{122}(x_1,x_2,x_3)]_{x_3=0}=0.$$

The solution of the constraint equations fixes the eigenvalue and the state. The ground-state eigenvalue is given by $E_g(0,1,-1;N=3)=\frac{\hbar^2}{2m}(\frac{2\pi}{L})^2[(\hat{\varphi})^2+(1-\hat{\varphi})^2+(-1-\hat{\varphi})^2]$ and the three particles ground state is:

$$\begin{aligned}|0,1,-1;N=3\rangle=&\int_0^L dx_1\int_0^L dx_2\int_0^L dx_3\\&\times\Bigg[\Phi_{0,1,-1}(x_1,x_2,x_3)C_1^\dagger(x_2)C_1^\dagger(x_2)C_1^\dagger(x_3)\\&+3\Bigg\{\epsilon_{1,1;2}^{2,1}\sum_{i=x_1,x_2,x_3}\hat{P}_{i,x_3}[\Phi_{0,1}(x_1,x_2)Z(x_3)\\&-\Phi_{0,-1}(x_1,x_2)Z^*(x_3)]\\&+\Phi_{0,1,-1}(x_1,x_2,x_3)\Bigg\}C_1^\dagger(x_1)C_1^\dagger(x_2)C_2^\dagger(x_3)\Bigg]\\&+3\Bigg[\Bigg\{\epsilon_{1;2,2}^{1,2}\sum_{i=x_1,x_2,x_3}\hat{P}_{i,x_1}[\Phi_{0,1}(x_2,x_3)Z(x_1)\\&-\Phi_{0,-1}(x_2,x_3)Z^*(x_1)]\\&+\Phi_{0,1,-1}(x_1,x_2,x_3)\Bigg\}C_1^\dagger(x_1)C_2^\dagger(x_2)C_2^\dagger(x_3)\Bigg]\\&+\Phi_{0,1,-1}(x_1,x_2,x_3)C_2^\dagger(x_1)C_2^\dagger(x_2)C_2^\dagger(x_3)]|0\rangle.\end{aligned}\tag{17}$$

This state is expressed in terms of the Slater determinants for two and three particles $\Phi_{0,\pm1}(x_1,x_2)$, $\Phi_{0,1,-1}(x_1,x_2,x_3)$. Here $\hat{P}_{i,x_3}$ is the coordinates interchange operator defined by:

$$\hat{P}_{i,x_3}F(x_1,x_2;x_3)=\delta_{i,x_3}F(x_1,x_2;x_3)+\delta_{i,x_1}F(x_3,x_2;x_1)+\delta_{i,x_2}F(x_1,x_3;x_2).$$

The three particles states can be rewritten as an antisymmetric tensor product of the three single particles states, which obey Eq. (11):

$$|0,1,-1;N=3\rangle=\sum_P(-1)^P|0_{P(1)};N=1\rangle|1_{P(2)};N=1\rangle\times|-1_{P(3)};N=1\rangle.$$

D. Four particles state

The wave function for *four* particles has the structure $|n,-n,m,-m;N=4\rangle$ with $n\neq m$. The ground state is given by: $|1,-1,2,-2;N=4\rangle$ with the eigenvalue $E_g(1,-1,2,-2;N=4)$. From Eq. (8) we find: $H|1,-1,2,-2;N=4\rangle=E(4)|1,-1,2,-2;N=4\rangle$, $\eta|1,-1,2,-2;N=4\rangle=0$, $E|1,-1,2,-2;N=4\rangle=0$, and $\gamma|1,-1,2,-2;N=4\rangle=0$; we obtain a set of equations for the spinor components $f_{1111}(x_1,x_2,x_3,x_4)$, $f_{1112}(x_1,x_2,x_3,x_4)$, $f_{1122}(x_1,x_2,x_3,x_4)$, $f_{1222}(x_1,x_2,x_3,x_4)$, and $f_{2222}(x_1,x_2,x_3,x_4)$:

$$4f_{1111}(x_1,x_2,x_3,0)=f_{1112}(x_1,x_2,x_3,0);$$

$$[4\partial_{x_4}f_{1111}(x_1,x_2,x_3,x_4)+\partial_{x_4}f_{1112}(x_1,x_2,x_3,x_4))]_{x_4=0}=0,$$

$$4f_{2222}(x_1,x_2,x_3,0)=-f_{1222}(0,x_1,x_2,x_3);$$

$$[4\partial_{x_4}f_{2222}(x_1,x_2,x_3,x_4)-\partial_{x_4}f_{1222}(x_1,x_2,x_3,x_4)]_{x_4=0}=0,$$

$$3f_{1112}(x_1,x_2,0,x_4)=-2f_{1122}(x_1,x_2,x_3,0);$$

$$[3\partial_{x_4}f_{1112}(x_1,x_2,x_3,x_4)-2\partial_{x_4}f_{1122}(x_1,x_2,x_3,x_4)]_{x_4=0}=0,$$

$$3f_{1222}(x_1,x_2,x_3,0)=-2f_{1221}(x_1,x_2,x_3,0);$$

$$[3\partial_{x_4}f_{1222}(x_1,x_2,x_3,x_4)+2\partial_{x_4}f_{1221}(x_1,x_2,x_3,x_4)]_{x_4=0}=0.$$

The eigenvalue and the eigenfunction are

$$\begin{aligned}&E_g(1,-1,2,-2;N=4)\\&=\frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2[(1-\hat{\varphi})^2+(-1-\hat{\varphi})^2\\&+(2-\hat{\varphi})^2+(-2-\hat{\varphi})^2],\end{aligned}$$

$$\begin{aligned}|1,-1,2,-2;N=4\rangle=&\int_0^L dx_1\int_0^L dx_2\int_0^L dx_3\int_0^L dx_4\Bigg[\Phi_{1,-1,2,-2}(x_1,x_2,x_3,x_4)C_1^\dagger(x_1)C_1^\dagger(x_4)\\&\times C_1^\dagger(x_3)C_1^\dagger(x_4)+4\epsilon_{1,1,1;2}^{3,1}\Bigg[\sum_{i=x_1,x_2,x_3,x_4}\hat{P}_{i,x_4}\{\Phi_{2,1,-1}(x_1,x_2,x_3)[Z(x_4)]^2-\Phi_{-2,1,-1}(x_1,x_2,x_3)[Z^*(x_4)]^2\\&+\Phi_{1,2,-2}(x_1,x_2,x_3)Z(x_4)-\Phi_{-1,2,-2}(x_1,x_2,x_3)Z^*(x_4)\}\Bigg]C_1^\dagger(x_1)C_1^\dagger(x_2)C_1^\dagger(x_3)C_2^\dagger(x_4)+6\epsilon_{1,1;2,2}^{2,2}\Bigg\{\Bigg(\sum_{i=x_1,x_2,x_3}\hat{P}_{i,x_3}\\&+\sum_{i=x_1,x_2,x_4}\hat{P}_{i,x_4}\Bigg)[\Phi_{1,-1}(x_1,x_2)\Phi_{2,-2}(x_3,x_4)+\Phi_{1,2}(x_1,x_2)\Phi_{-1,-2}(x_3,x_4)]\Bigg\}C_1^\dagger(x_1)C_1^\dagger(x_2)C_2^\dagger(x_3)C_2^\dagger(x_4)\end{aligned}$$

$$+4\epsilon^{1,3}_{1;2,2,2}\Bigg[\sum_{i=x_1,x_2,x_3,x_4}\hat{P}_{i,x_1}\{\Phi_{2,1,-1}(x_2,x_3,x_4)[Z(x_1)]^2-\Phi_{-2,1,-1}(x_2,x_3,x_4)[Z^*(x_1)]^2+\Phi_{1,2,-2}(x_2,x_3,x_4)Z(x_1)$$
$$-\Phi_{-1,2,-2}(x_2,x_3,x_4)Z^*(x_1)\}\Bigg]C_1^\dagger(x_1)C_2^\dagger(x_2)C_2^\dagger(x_3)C_2^\dagger(x_4)+\Phi_{1,-1,2,-2}(x_1,x_2,x_3,x_4)C_2^\dagger(x_1)C_2^\dagger(x_2)C_2^\dagger(x_3)C_2^\dagger(x_4)\Bigg]$$
$$\times|\times 0\rangle\equiv\sum_P(-1)^P|1_{P(1)};N=1\rangle|-1_{P(2)};N=1\rangle|2_{P(3)};N=1\rangle|-2_{P(4)};N=1\rangle. \tag{18}$$

Where $\Phi_{1,-1,2,-2}(x_1,x_2,x_3,x_4)$, $\Phi_{\pm 2,1,-1}(x_1,x_2,x_3)$, and $\Phi_{n,m}(x_1,x_2)$ are the Slater determinants for 2, 3, and 4 particles, respectively. Here $\epsilon^{3,1}_{1,1,1;2}$ and $\epsilon^{2,2}_{1,1;2,2}$ are the antisymmetric tensors for the ring index.

E. 2*M* particles state

The $2M$ particles state is built from the single particles states $n_1,..n_k,..n_M$ given by Eq. (11) with vanishing total momentum:

$$|n_1,-n_2,\ldots n_{2k-1},-n_{2k},\ldots n_{2M-1},-n_{2M};N=2M\rangle$$
$$=\sum_P(-1)^P|n_{P(1)};N=1\rangle|-n_{P(2)};N=1\rangle\ldots|n_{P(2M-1)};N=1\rangle$$
$$\times|-n_{P(2M)};N=1\rangle. \tag{19}$$

The ground state and the ground-state energy are

$$|1,-1,\ldots M,-M;N=2M\rangle_g=\Sigma_P(-1)^P|1_{P(1)};N=1\rangle$$
$$\times|-1_{P(2)};N=1\rangle|2_{P(3)};N=1\rangle$$
$$\times|-2_{P(4)};N=1\rangle\ldots|k_{P(2k-1);N=1}\rangle$$
$$\times|-k_{P(2k)};N=1\rangle\ldots|M_{P(2M-1)};N=1\rangle$$
$$\times|-M_{P(2M)};N=1\rangle$$

and

$$E_g(1,-1,\ldots,k,-k,\ldots M,-M)$$
$$=\frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2\Sigma_{k=1}^M[(k-\hat{\varphi})^2+(-k-\hat{\varphi})^2].$$

F. Current for equal fluxes

The current for equal fluxes with 1, 2, 3, 4, and $2M$ particles is the same in both rings:

$$J_1^{N=1}=\frac{\langle N=1;n|\hat{J}_1(x)|n;N=1\rangle}{\langle N=1;n|n;N=1\rangle}=\left[\frac{\hbar}{m}\frac{2\pi}{L}\right]\left[\frac{\hat{\varphi}-n}{2L}\right];\quad n=0,\pm 1,\pm 2\ldots$$

$$J_1^{N=1}\left(\hat{\varphi}=\frac{1}{2}\right)=\frac{\left\langle N=1;\hat{\varphi}=\frac{1}{2}\middle|\hat{J}_1(x)\middle|\hat{\varphi}=\frac{1}{2};N=1\right\rangle}{\left\langle N=1;\hat{\varphi}=\frac{1}{2}\middle|\hat{\varphi}=\frac{1}{2};N=1\right.}=\left[\frac{\hbar}{m}\frac{2\pi}{L}\right][|\alpha_+|^2-|\alpha_-|^2]\left[\frac{\hat{\varphi}-n}{2L}\right],$$

$$J_1^{N=2}=\frac{\langle N=2;-1,1|\hat{J}_1(x)|1,-1;N=2\rangle}{\langle N=2;-1,1|1,-1;N=2\rangle}=\left[\frac{\hbar}{m}\frac{2\pi}{L}\right]\left[\frac{2\hat{\varphi}}{2L}\right],$$

$$J_1^{N=3}=\frac{\langle N=3;-1,1,0|\hat{J}_1(x)|0,,1,-1;N=3\rangle}{\langle N=3;-1,1,0|0,1,-1;N=3\rangle}=\left[\frac{\hbar}{m}\frac{2\pi}{L}\right]\left[\frac{3\hat{\varphi}}{2L}\right],$$

$$J_1^{N=4}=\frac{\langle N=4;-2,2,-1,1|\hat{J}_1(x)|1,-1,2,-2;N=4\rangle}{\langle N=4;-2,2,-1,1|1,-1,2,-2;N=4\rangle}=\left[\frac{\hbar}{m}\frac{2\pi}{L}\right]\left[\frac{4\hat{\varphi}}{2L}\right],$$

$$J_1^{N=2M}=\frac{\langle N=2M;-M,M,\ldots-1,1|\hat{J}_1(x)|1,-1,\ldots M,-M;N=2M\rangle_g}{\langle N=2M;-M,M,\ldots-1,1|1,-1,\ldots M,-M;N=2M\rangle_g}=\left[\frac{\hbar}{m}\frac{2\pi}{L}\right]\left[\frac{2M\hat{\varphi}}{2L}\right]. \tag{20}$$

The magnetization $M^{(N)}$ is given by the product of current and area: $M^{(N)}=2J_1^N\frac{L^2}{4\pi}$. For an even number of electrons we find that the current in a single ring is twice the current in a double ring $J^{N=2M}_{\text{single-ring}}=2J_1^{N=2M}$. The factor of $\frac{1}{2}$ is a result of the two component *spinor* state renormalization. At finite temperatures the two rings excited states have the form: $|1,-1,\ldots M+p,-(M+p);N=2M\rangle_e$ where p are integers. This state carries the same current as the ground state $|1,-1,\ldots M,-M;N=2M\rangle_g$. Therefore, we conclude that for an *even* (fixed) number of electrons the current will be the same at any temperature. (When the total number of electrons fluctuates, $N\to N\pm 2$, thermal effects will decrease the current.) The situation for the *odd* number of electrons is different. Even for the two states $|1,-1,\ldots M,-M,n=(M+p);N=2M+1\rangle$ and $|1,-1,\ldots M,-M,n=-(M+p);N=2M+1\rangle$ we have different eigenvalues and at finite temperatures these states carry a different current. Therefore, the total current carried by all the states will be reduced like we have for a single ring where the unrestricted structure of the wave function allows any configuration of momenta, which generate an antisymmetric wave function in space: $f^{(\text{single-ring})}(x_1,x_2,\ldots x_{N=2M})=\Phi_{n_1,n_2,\ldots,n_{2M}}(x_1,x_2,\ldots x_{N=2M})$. To probe this even-odd structure experimentally we propose to attach a gate (voltage) to the rings. As a result, the magnetization will vary with the varying gate voltage.

V. WAVE FUNCTION FOR OPPOSITE FLUXES

For this case the constraint operator γ is modified to: $\gamma=\{(-i\partial_x-\frac{2\pi}{L}\hat{\varphi})[C_1(x)+C_2(x)]\}|_{x=0,L}$. For the single particle case we find the following boundary conditions:

$$f_1(x=0)=f_2(x=0);$$

$$-i[\partial_x f_1(x)+\partial_x f_2(x)]|_{x=0,L}=\frac{2\pi}{L}\hat{\varphi}[f_1(x)+f_2(x)]|_{x=0,L}.$$

We find that for this case the wave function must vanish. Only for integer values of flux $n=\text{integer}=\hat{\varphi}$ we have finite solutions $f_1(x)=f_2(x)=e^{i(2\pi n/L)x}$ with a vanishing persistent current. This result is in agreement with the fact that at the common point between the rings the fluxes must satisfy $\hat{\varphi}_2=\hat{\varphi}_1+n$. Therefore, the boundary condition can be satisfied for this case only if the wave function vanishes at the common point. We mention that for two separated rings threaded by opposite fluxes the magnetization will be zero only at the symmetry points. This result allows one to control the current in one ring by reversing the flux in the second ring.

VI. TWO COUPLED CYLINDERS

In order to build a realistic theory which can be compared with experiments we have to consider effects of interactions and effects of finite width geometry.[12] For realistic considerations the point contact between the two rings is replaced by two *narrow cylinders* of height $d\ll L$, which are in contact at the point $x=0,0\le z\le d$ shown in Fig. 3. The gluing condition is implemented by two narrow cylinders of height d replacing the constraints in Eq. (6) by $\eta(z,x=0)|\chi,N\rangle=0$ and $\gamma(z,x=0)|\chi,N\rangle=0$. In the absence of disorder we obtain for each transversal channel $r=1,2,\ldots,r_{\max}$ one dimensional constraints: $\eta_r(x=0)|\chi,N\rangle=0$; $\gamma_r(x=0)|\chi,N\rangle=0$.

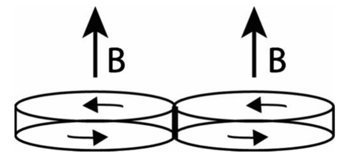

FIG. 3. Two coupled cylinders of height d with a flux. The coordinate y is defined along the rim of the cylinders similar to Fig. 1.

Therefore, the current in the channel r is the same as the result given in Eq. (20). For N electrons the current will be determined by the partition of N electrons in the different channels: $N=N_1+N_2+\ldots+N_r+\ldots+N_{r_{\max}}$. In the absence of disorder the current in cylinder one, at T=0, will be given by:

$$J_1^N=\left[\frac{\hbar}{m}\frac{2\pi}{L}\right]\left[\frac{2(N_1+N_2+\;.\;.\;N_r+\;\ldots N_{r_{\max}})\hat{\varphi}}{2L}\right]\equiv\left[\frac{\hbar}{m}\frac{2\pi}{L}\right]\left[\frac{2N\hat{\varphi}}{2L}\right].$$

VII. CONCLUSION

To conclude, the wave function for coupled rings in the presence of constraints has been computed using a formalism introduced earlier (arXiv:0907.2458). This method has been used to compute the wave function for coupled rings invoking a folded geometry and a spinor representation. For an even number of electrons, only states with total vanishing momentum are allowed giving rise to a large persistent current and magnetization. For odd number of electrons at finite temperature the current and the magnetization are suppressed. We propose to confirm this even-odd effect experimentally by attaching the two rings to a varying gate voltage. Reversing the flux in one ring will cause the current to vanish in both rings. We construct the many particle ground state, which obeys the constraints and show that not all the many particle states which are built from single particle states (which also obey the constraints) are allowed. There are potential implications of these results in quantum entanglement and quantum information processing, e.g., coupled rings as a nonlocal control device or qubits. Finally, we note that a related two-ring problem has been considered in Ref. 13.

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